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Conserved charges in the principal chiral model on a supergroup

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Abstract: The classical principal chiral model in 1+1 dimensions with target space a compact Lie supergroup is investigated. It is shown how to construct a local conserved charge given an invariant tensor of the Lie superalgebra. We calculate the super-Poisson brackets of these currents and argue that they are finitely generated. We show how to derive an infinite number of local charges in involution. We demonstrate that these charges Poisson commute with the non-local charges of the model.

Keywords: Superspaces, Global Symmetries, Sigma Models, Integrable Field Theories.

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Contents

1. Introduction

Finding exact methods of solving string theory on AdS space, as motivated by the AdS/CFT correspondence, remains a difficult problem. There are two schemes for describing such a superstring. The NSR description gives a free action in a flat background but the existence of the RR vertex operators introduces seemingly insurmountable difficulties. Alternatively there is the GS formalism in which the supersymmetry exists on the target manifold which is then described as a Lie supergroup or supercoset space.

It has proved possible to describe superstring theory on $AdS_n \times S^n$ as a coset space G/H (for example, $AdS_2 \times S^2$ is the bosonic subalgebra of $\frac{\mathrm{PSU}(1,1|2)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ [1], and $AdS_5 \times S^5$ is the same for $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ [2 − 4].) Although G is Ricci flat in both these instances, the coset space is not. However, H is the invariant locus of a $\mathbb{Z}_4$ automorphism, and this permits the introduction of a WZ-term to provide a quantum conformal theory. (Indeed [5, 6] have demonstrated quantum conformal invariance given a $\mathbb{Z}_n$ automorphism.) Similarly superstring theory on $AdS_3 \times S^3$ is related to a sigma model on $\mathrm{PSU}(1,1|2)$ [7]. More general work has looked at sigma models on the supergroups $\mathrm{PSL}(n|n)$ [8].

All of these models contain an infinite number of local and non-local conserved charges. Their existence constrains the S-matrix and permits its exact computation. These charges have been studied in both bosonic and worldsheet supersymmetric principal chiral models (PCMs) [9], and for sigma models on symmetric spaces [10, 11].

The PCM on any Lie supergroup is classically conformal. In the quantum model, the one loop beta function is proportional to the dual Coxeter number $h^\vee$ [8, 12], and there are some superalgebras for which this vanishes (namely $psu(n|n)$ and $osp(2n+2|2n)$). The PCM on these supergroups will therefore be quantum conformal [8, 13]. However we do not expect conformal invariance to survive in the quantum model for general supergroups when the model becomes massive. This paper is a modest attempt at understanding the algebra of these models' local charges, with the hope that some insight could be given to the perhaps more physical models.

Ultimately we find that the classical PCM on a supergroup has an infinite number of charges in involution. These charges are formed from integrals of local conserved currents, each associated to an invariant of the Lie superalgebra. Only a finite number of these are independent, and from these invariants it is possible to construct all conserved local currents.

For the $su(m|n)$ models we will construct a set of currents which give rise to commuting charges. The construction will fail for $m = n$, so we restrict ourselves to the massive cases. The situation is analogous for $osp(m|2n)$, except now a family of currents exists, depending on a free parameter α. For $osp(2m|2n)$ an additional invariant exists, the superpfaffian [14], and requiring that the superpfaffian charge commutes with the other local charges fixes the value of α, but only when the model is massive.

These charges are conserved classically, but anomalies might arise in the quantum model. However we anticipate that integrability survives because we can use Goldschmidt and Witten's method of anomaly counting [15] to show that higher spin conservation laws still exist.

In addition to the local charges, there are non-local charges which form a Super-Yangian structure [16 – 18]. We shall show that these non-local charges are in involution with the local charges.

1.1 Lie superalgebras

We begin with a basic introduction to superalgebras, and refer to the literature [19 – 21] for a more thorough review. Throughout this paper, letters $A, B, C, \dots$ will denote both bosonic and fermionic indices. $a, b, c, \dots$ will be used for bosonic indices, and $\alpha, \beta, \gamma, \dots$ for fermionic indices.

We begin with an associative Grassmann algebra $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ with sufficiently many anticommuting generators, where $\Lambda_{\bar{0}}$ (resp. $\Lambda_{\bar{1}}$) consists of commuting (resp. anticommuting) elements. We have the product rule $\Lambda_{\bar{i}} \cdot \Lambda_{\bar{j}} \subset \Lambda_{\overline{i+j}}$. (Addition modulo 2 is left implicit here and throughout.)

Given a supermatrix $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ we define it to be even (odd) if $A, D \in \Lambda_{\bar{0}}$ ($\Lambda_{\bar{1}}$) and $B, C \in \Lambda_{\bar{1}}$ ($\Lambda_{\bar{0}}$). We write $\deg(X) = 0$ if it is even and $\deg(X) = 1$ if odd. We can then define the supertrace as

$$\mathrm{Str}(X) = \mathrm{Tr}(A) - (-1)^{\deg(X)}\mathrm{Tr}(D) \tag{1.1}$$

the supertranspose as

$$X^{ST} = \begin{pmatrix} A^T & -(-1)^{\deg(X)} B^T \\ (-1)^{\deg(X)} C^T & D^T \end{pmatrix} \tag{1.2}$$

and if X is even and invertible, the superdeterminant as

$$\mathrm{sdet}(X) = \frac{\det(A - BD^{-1}C)}{\det(D)} = \frac{\det(A)}{\det(D - CA^{-1}B)}. \tag{1.3}$$

These satisfy the important properties

$$(XY)^{ST} = (-1)^{\deg(X)\deg(Y)} Y^{ST} X^{ST} \tag{1.4}$$

$$\mathrm{Str}(XY) = (-1)^{\deg(X)\deg(Y)} \mathrm{Str}(YX) \ , \ \mathrm{Str}(X^{ST}) = \mathrm{Str}(X) \tag{1.5}$$

$$\mathrm{sdet}(XY) = \mathrm{sdet}(X)\mathrm{sdet}(Y) \ , \qquad \mathrm{sdet}(X^{ST}) = \mathrm{sdet}(X) \tag{1.6}$$

$$\mathrm{sdet}(\exp(X)) = \exp(\mathrm{Str}(X)) \tag{1.7}$$

We consider a simple connnected supergroup G [19], either $\mathrm{SU}(m|n)$ for $m \neq n$, or the compact subgroup of $\mathrm{OSp}(m|2n)$ which is connected to the identity, satisfying

$$\mathrm{SU}(m|n) \ : \ \mathrm{sdet}X = 1 \quad XX^\dagger = 1 \tag{1.8}$$

$$\mathrm{OSp}(m|2n) \ : \mathrm{sdet}(X) = 1 \ X^{ST} H X = H \tag{1.9}$$

where $H = \left(\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right)$ for an symmetric I and symplectic J. There exists a basis for which I is the identity $\mathbb{I}_m$ and $J = \left(\begin{smallmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{smallmatrix}\right)$.

Instead of the non-simple Lie superalgebra $\mathrm{SU}(n|n)$, we shall look at $\mathrm{PSU}(n|n)$, which is $\mathrm{SU}(n|n)$ modulus the identity.

Given an element of the supergroup, we can write it in terms of its superalgebra generators T^A

$$g = \exp(x_A T^A) = \exp(x_a T^a) \exp(x_\alpha T^\alpha) \tag{1.10}$$

where T^a (respectively T^α) generate $\mathbf{g}_{\bar{0}}$ (respectively $\mathbf{g}_{\bar{1}}$). These satisfy the supercommutation relationship

$$[T^A, T^B] = T^A T^B - (-1)^{\eta_A \eta_B} T^B T^A = f^{AB}{}_C T^C \tag{1.11}$$

for structure constants $f^{AB}{}_C$, anti-supersymmetric in the first two, and the first and last indices.

We denote by η_A the grade of T^A in the Lie superalgebra. The x_A commute (resp. anticommute) whenever the T^A are graded even (resp. odd), and so we can define without ambiguity η_A to be the gradation of x_A in the Grassmann algebra. The local currents from the PCM are constructed from elements of this Lie algebra.

We are interested in the Lie superalgebras $su(m|n)$, and $osp(m|2n)$, satisfying

$$su(m|n) : \mathrm{Str}(X) = 0 \ , \ X = -X^\dagger \tag{1.12}$$

$$osp(m|2n) : X^{ST} = -HXH^{-1} \tag{1.13}$$

For $psu(n|n)$ we begin with $gl(n|n)$ and its generators: the $2n \times 2n$ identity $\mathbb{I}$, $\mathbb{J} = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$, and the supertraceless, traceless and anti-hermitian T^A, $A = 1, \dots, 4n^2 - 2$. The identity and the T^A generate $sl(n|n)$. The T^A get projected onto the generators of $psu(n|n)$, but do not close under commutation

$$[T^A, T^B] = F^{AB}_{\ \ C} T^C + d^{AB}\mathbb{I}$$

and so we always work modulus the identity. i.e. Two elements $X, Y \in psu(n|n)$ are equivalent if their difference is a multiple of the identity.

For the Lie superalgebras $su(m|n)$ $(m \neq n)$, $psu(n|n)$ and $osp(m|2n)$ we can define a non-degenerate invariant bilinear form $G^{AB} = \mathrm{Str}(T^A T^B)$. Note that this implies that $G^{AB} = 0$ unless $\eta_A = \eta_B$ (consistency) and that $G^{AB} = (-1)^{\eta_A} G^{BA}$ (supersymmetry). We define the inverse G_{AB} via

$$G_{AB} G^{BC} = \delta_A^C \tag{1.14}$$

Note in particular that this implies that

$$G^{AB} X_A Y_B = X_A Y^A = (-1)^{\eta_A} X^A Y_A = G_{BA} X^A Y^B. \tag{1.15}$$

We will make much use of the completeness condition in what is to follow. For any $X = X_A T^A \in \mathbf{g}$ we have

$$X_A = \mathrm{Str}(T_A X). \tag{1.16}$$

2. Local charges of the PCM

The PCM is defined by the lagrangian

$$\mathcal{L} = \frac{\kappa}{2} \mathrm{Str}(\partial_\mu g^{-1} \partial^\mu g) \tag{2.1}$$

where g takes values in a supergroup G, either $\mathrm{SU}(m|n)$ or $\mathrm{OSp}(m|2n)$. κ is a dimensionless constant, its value unimportant for the classical model, and μ indexes the 1+1D spacetime. This lagrangian is invariant under a global chiral symmetry $g(x,t) \to g_1 g(x,t) g_2^{-1}$, with associated Noether conserved local currents

$$j^L_\mu = \kappa \partial_\mu g g^{-1} \quad , \quad j^R_\mu = -\kappa g^{-1} \partial_\mu g. \tag{2.2}$$

The conservation of these currents are the equations of motion. These currents belong to the even subspace of $\mathbf{g} \otimes \Lambda$.

The $\mathrm{PSU}(n|n)$ model is more complicated because no matrix representation for the superalgebra exists. It has been suggested [8] to consider the $\mathrm{SU}(n|n)$ model, which is in addition invariant under the U(1) gauge symmetry $g(x) \mapsto e^{\phi(x)} g(x)$. However no non-degenerate metric exists for $su(n|n)$, and this will complicate the construction of commuting charges. It would be interesting to pursue the construction further because of the quantum conformal invariance of these models.

From the Noether conserved currents (2.2) it is possible to construct higher spin local and non-local conserved charges. Using either the left or right currents will give rise to identical local charges. The non-local charges constructed from them are not equal, but form two copies of a super-Yangian structure. When there is no confusion, we shall drop the L/R indices.

We assume appropriate boundary conditions on $j_\mu(x)$

$$j_\mu(x) = 0 \quad \text{as } x \to \pm\infty. \tag{2.3}$$

The currents are conserved, and satisfy the Bianchi identity

$$\partial^\mu j_\mu = 0 \quad , \quad \partial_\mu j_\nu - \partial_\nu j_\mu - \frac{1}{\kappa}[j_\mu, j_\nu] = 0. \tag{2.4}$$

Immediately we can form conserved charges out of the Noether currents (2.2) by $Q^{(0)} = \int dx\ j_0$. More importantly though, conditions (2.4) allow us to form a local conserved charge of spin s through the use of a G-invariant tensor of degree $s+1$. Although there are infinitely many such invariant tensors, there are only finitely many independent (primitive) tensors. The number is equal to the rank of $\mathbf{g}$. All other invariants are constructed from these primitive ones.

The intention is to find a maximal set of mutually commuting conserved local charges $\{q_s\}$. The existence of these currents displays the integrable nature of the PCM, because (equivalent to the construction of non-local charges) the two conditions in (2.4) allow a Lax pair to be constructed.

It will prove useful to write conditions (2.4) in terms of light-cone coordinates, $x^\pm = \frac{1}{2}(t \pm x)$

$$\partial_- j_+ = -\partial_+ j_- = -\frac{1}{2\kappa}[j_+, j_-]. \tag{2.5}$$

2.1 The energy-momentum tensor and conformal invariance

The energy-momentum tensor is the variation of the lagrangian with respect to the space-time metric,

$$T_{\mu\nu} = -\frac{1}{2\kappa}\left(\mathrm{Str}(j_\mu j_\nu) - \frac{1}{2}\eta_{\mu\nu}\mathrm{Str}(j_\rho j^\rho)\right) \tag{2.6}$$

This is traceless, symmetric and conserved. In light-cone coordinates

$$T_{\pm\pm} = -\frac{1}{2\kappa}\mathrm{Str}(j_\pm j_\pm)\ , \quad T_{+-} = T_{-+} = 0. \tag{2.7}$$

and

$$\partial_- T_{++} = \partial_+ T_{--} = 0 \tag{2.8}$$

Here T_{+-} is the trace of the two-dimensional energy-momentum and its vanishing implies that the model has classical conformal invariance. The situation is more complicated in the presence of quantum anomalies. The one loop beta function is proportional to $h^\vee$, the dual Coxeter number of the Lie (super)algebra [12]. For purely bosonic Lie algebras, $h^\vee \neq 0$, and so a WZ term must be added to the lagrangian for conformal invariance in the quantum model [9]. However there are some Lie superalgebras (namely $psl(n|n)$ and $osp(2n+2|2n)$) for which $h^\vee = 0$, and these models retain conformal invariance in the quantum model, at least to one loop [8, 13].

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We note here that we can form a series of higher-spin conservation laws,

$$\partial_-(T^p_{++}) = \partial_+(T^p_{--}) = 0. \tag{2.9}$$

These give the classical conformal symmetry of the model, but are not expected to be preserved for the quantum models for which the dual Coxeter number is non-zero. We shall not be concerned with these directly however, as we shall see that more general higher-spin currents can be formed.

2.2 Canonical formalism

Our aim here is to calculate the super-Poisson brackets (SPBs) for the current components $j_{\mu A}$ [22]. As in the bosonic case [9, 23], it is convenient to introduce the non-local operator $\Delta_1 = \partial_1 - \frac{1}{\kappa}[j_1, \ ]$ using which the Bianchi identity is re-expressed as $j_0 = \Delta_1^{-1}(\partial_0 j_1)$. We can now write the action as a functional of $j_1(x)$ only

$$\mathcal{L} = \frac{1}{2\kappa}\mathrm{Str}((\Delta_1^{-2}\partial_0 j_1)(\partial_0 j_1) - j_1^2) \tag{2.10}$$

(where we have imposed suitable boundary conditions such that $\Delta_1^{-1}(A)B = -A\Delta_1^{-1}(B)$ up to a total divergence).

Defining the conjugate momentum of j_1 to be $\pi = \pi^A T_A$, where $\pi^A = \partial\mathcal{L}/\partial(\partial_0 j_{1A})$ we find that $j_0 = -2\kappa\Delta_1\pi$. Then, using $\{j_{1A}(x), \pi^B(y)\} = \delta_A^B\delta(x-y)$, we find that

$$\begin{aligned}
\{j_{0A}(x), j_{0B}(y)\} &= (-1)^\chi f_{AB}{}^C j_{0C}(x)\delta(x-y) \\
\{j_{0A}(x), j_{1B}(y)\} &= (-1)^\chi f_{AB}{}^C j_{1C}(x)\delta(x-y) \\
&\quad +\kappa G_{AB}\partial_x\delta(x-y) \\
\{j_{1A}(x), j_{1B}(y)\} &= 0
\end{aligned} \tag{2.11}$$

where $\chi = \eta_A.\eta_B + \eta_A + \eta_B$. In light-cone coordinates these become

$$\begin{aligned}
\{j_{\pm A}(x), j_{\pm B}(y)\} &= (-1)^\chi f_{AB}{}^C\left(\frac{3}{2}j_{\pm C}(x) - \frac{1}{2}j_{\mp C}(x)\right)\delta(x-y) \\
&\quad \pm 2\kappa G_{AB}\delta'(x-y) \\
\{j_{+A}(x), j_{-B}(y)\} &= \frac{1}{2}(-1)^\chi f_{AB}{}^C\left[j_{+C}(x) + j_{-C}(y)\right]\delta(x-y)
\end{aligned} \tag{2.12}$$

These are very similar in form to the Poisson brackets of the bosonic model [9] but we must take account of the non-trivial gradings.

2.3 Higher-spin conserved charges

We can construct the Noether $G_L \times G_R$ conserved charges

$$Q_A^0 = \int_{-\infty}^{\infty} dx\, j_{0A}(x) \tag{2.13}$$

for both left and right currents j^L_μ and j^R_μ. More interestingly we can use the above SPBs to find an infinite set of commuting higher-spin holomorphic local currents. We shall obtain the same set of local currents if we use either the left or right current. We first note that, similar to the bosonic model [9], to every invariant tensor $d^{A_1\dots A_p}$ (supersymmetric in adjacent indices) associated with a Casimir element of degree p, we can associate a conserved current of spin p. Denote such a Casimir element

$$\mathcal{C}^p = d^{A_1,\dots A_p} T_{A_1} \dots T_{A_p} \tag{2.14}$$

and supersymmetry means that

$$d^{A_1\dots A_k A_{k+1}\dots A_p} = (-1)^{\eta_{A_k}\eta_{A_{k+1}}} d^{A_1\dots A_{k+1} A_k \dots A_p} \tag{2.15}$$

We note that, for $su(m|n)$ and $osp(m|2n)$, $d^{A_1,\dots A_p} = 0$ unless $\sum \eta_{A_i} = 0$. Indeed, Casimir elements are bosonic for all basic superalgebras except $Q(n)$ [19]. Invariance then implies that

$$[\mathcal{C}^p, T^B] = 0 \quad \Rightarrow \quad \sum_{i=1}^{p} (-1)^{\eta_B(\eta_{A_i}+\cdots \eta_{A_p})} d^{A_1\dots \hat{A}_i C \dots A_p} f^{A_i B}{}_C = 0 \tag{2.16}$$

We define the action of this tensor on an element of the Lie superalgebra $X = X_A T^A$ by

$$d^{(p)}(X) = d^{A_1\dots A_p} X_{A_1} \dots X_{A_p} \tag{2.17}$$

and observe that it is G-invariant

$$d^{(p)}(gXg^{-1}) = d^{(p)}(X) \text{ for } g \in G. \tag{2.18}$$

We can simplify matters by noting that every element of a Lie (super)algebra is locally conjugate to some element of its Cartan subalgebra (CSA), $\mathbf{h} \subset \mathbf{g}$ i.e. its maximal abelian subgroup. For most basic Lie superalgebras, $\mathbf{h}$ is the CSA of its bosonic subalgebra $\mathbf{h} \subset \mathbf{g}_{\bar{0}}$.

Using the invariance property we then have

$$d^{(p)}(X) = d^{(p)}(H) = d^{a_1\dots a_p} H_{a_1} \dots H_{a_p} \tag{2.19}$$

where $H = H_a T^a = gXg^{-1} \in \mathbf{h}$. We can thus consider the invariant tensor to be restricted to the CSA, and so we are only interested in the invariants of the underlying bosonic Lie subalgebra.[1]

To each superalgebra there are an infinite number of invariant tensors, notably of the form $\mathrm{Str}(T^{A_1} \dots T^{A_m})$, but there are only finite many independent (or primitive) tensors, the number being equal to the rank of the superalgebra. All local conserved charges will be generated by charges formed from these primitive tensors. These primitive tensors were discussed in [24, 25] for the bosonic algebras, and for superalgebras in [26, 27]

An infinite set of higher spin conserved charges can be constructed from tensors which satisfy (2.16). Using (2.5) it is straightforward to verify that

$$\partial_-(d^{A_1\dots A_p} j_{+A_p} \dots j_{+A_1}) = 0 \tag{2.20}$$

[1]The exception is the strange superalgebra $Q(n)$, where $\mathbf{h} \cap \mathbf{g}_{\bar{1}} \neq \emptyset$ [19].

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where the change in ordering of the indices is non-trivial because of the $\mathbb{Z}_2$-grading (2.15). Then the local conserved charges are

$$q_{\pm s} = \int dx\, d^{A_1 \dots A_{s+1}} j_{\pm A_{s+1}}(x) \dots j_{\pm A_1}(x). \tag{2.21}$$

The charges are labelled by s, their spin. The Poisson bracket with the (purely bosonic) boost generator M is $\{M, q_{\pm s}\} = \pm s q_{\pm s}$. We shall define the charges $q_{\pm s}$ with $s > 0$ as having positive/negative chirality. The additive nature of these local charges implies that they have a trivial coproduct

$$\Delta(q_{\pm s}) = q_{\pm s} \otimes \mathbb{I} + \mathbb{I} \otimes q_{\pm s} \tag{2.22}$$

Using the invariance condition (2.16), it is simple to show that $\{q_{+r}, q_{-s}\} = 0$ for any integers $r, s > 0$. Furthermore, for charges of equal chirality, only the non-ultra local terms contribute.

$$\begin{aligned} \{q_{\pm r}, q_{\pm s}\} = \pm 2(r+1)(s+1)\kappa \int dx\, (-1)^{\eta_B(\eta_{B_1}+\dots+\eta_{B_s})} d^{A_1 \dots A_r A} d^{B_1 \dots B_s B} G_{AB} \\ \times j_{\pm A_1} \dots j_{\pm A_r} \partial_x (j_{\pm B_1} \dots j_{\pm B_s}) \end{aligned} \tag{2.23}$$

We are interested in the currents formed from a tensor with components $d^{A_1 \dots A_r} = \mathrm{sStr}(T^{A_1} \dots T^{A_r})$, which satisfies (2.16). sStr denotes the normalised supersymmetric supertrace

$$\mathrm{sStr}(T^{A_1} T^{A_2} \dots T^{A_r}) = \frac{1}{r!} \sum_{\sigma \in S_r} \epsilon_\sigma \mathrm{Str}(T^{A_{\sigma(1)}} T^{A_{\sigma(2)}} \dots T^{A_{\sigma(r)}}) \tag{2.24}$$

where S_r is the symmetric group of degree r, and $\epsilon_\sigma = -1$ if σ involves an odd number of permutations of fermionic indices, and equals 1 otherwise. This tensor is exactly zero unless $\eta_{A_1} + \dots + \eta_{A_r} = 0 \bmod 2$. This gives rise to holomorphic currents $\mathcal{J}_{\pm r} = \mathrm{Str}(j_\pm^r)$, and associated conserved charges

$$q_{\pm(r-1)} = \int dx\, \mathrm{Str}(j_\pm^r), \tag{2.25}$$

and we find that that equation (2.23) simplifies to

$$\{q_{\pm r}, q_{\pm s}\} = \pm 2(r+1)(s+1)\kappa \int dx\, \mathrm{Str}(j_\pm^r T_A) \partial_x \mathrm{Str}(j_\pm^s T^A). \tag{2.26}$$

We now want to find the currents $\mathcal{J}_r(x)$ (or indeed, algebraic functions of them) which give rise to charges in involution. In particular we note that $q_{\pm 2}$ always commutes with the other charges, showing that all higher-spin charges are classically in involution with energy-momentum.

We must now deal separately with the cases $G = \mathrm{SU}(m|n)$ and $G = \mathrm{OSp}(m|2n)$, and we shall see that in the former case we must impose $m \neq n$.

2.4 Commuting charges for SU$(m|n)$

We first consider the simple supergroups SU$(m|n)$ for $m \neq n$. It is evident that $X \in su(m|n)$ does not imply $X^p \in su(m|n)$ for all integers p because supertracelessness will not in general hold. In the case $m \neq n$ we can replace j_+^r by the supertraceless and anti-hermitian $j_+^r - (1/l)\mathrm{Str}(j_+^r)\mathbb{I}_{m+n}$ where $l = m - n$, and then using the completeness condition (1.16) we find

$$\{q_{\pm r}, q_{\pm s}\} = \mp\frac{2(r+1)(s+1)\kappa}{l}\int dx\, \mathrm{Str}(j_\pm^r)\partial_x \mathrm{Str}(j_\pm^s) \tag{2.27}$$

This term is not in general zero. It is necessary to know the exact form of the Poisson brackets for $\mathcal{J}_{\pm r} = \mathrm{Str}(j_\pm^r)$. After some computation the result is

$$\begin{aligned}\{\mathcal{J}_r(x), \mathcal{J}_s(y)\} = &\left(\frac{rs}{l}\mathcal{J}_{r-1}(x)\mathcal{J}_{s-1}(x) - rs\mathcal{J}_{r+s-2}(x)\right)\delta'(x-y) \\ &+ \left(\frac{rs}{l}\mathcal{J}_{r-1}(x)\mathcal{J}'_{s-1}(x) - \frac{rs(s-1)}{(r+s-2)}\mathcal{J}'_{r+s-2}(x)\right)\delta(x-y)\end{aligned} \tag{2.28}$$

Note the similarity between this and the Poisson brackets for the currents of the bosonic SU(l) model [9]. Indeed, these Poisson brackets are all antisymmetric, and feature purely bosonic currents. This is also the point where we must distinguish between the quantum conformal model PSU$(n|n)$ and the non-conformal models.

We now want to find a set of algebraically independent currents which give rise to a set of mutually commuting charges. We follow a similar method to [9] and define a generating function $A(x, \lambda)$ with a spectral parameter λ by

$$A(x,\lambda) = \mathrm{sdet}\,(1 - \lambda j_+(x)) = \exp\left(-\sum_{r=2}^{\infty}\frac{\lambda^r}{r}\mathcal{J}_r(x)\right) \tag{2.29}$$

and then claim that the set of currents defined by

$$\mathcal{K}_{r+1}(x) = A(x,\lambda)^{r/l}\Big|_{\lambda^{r+1}} \tag{2.30}$$

form commuting charges upon integration over space.

$$\int dx\, dy \{A(x,\mu)^{r/l}, A(y,\nu)^{s/l}\}\Big|_{\mu^{r+1}\nu^{s+1}} = 0 \tag{2.31}$$

This differs from the bosonic model [9] in that (2.29) is not a polynomial of finite order in λ (as it is in the bosonic case), but a rational function. For the bosonic model, when $r \equiv 0 \bmod l$ current (2.30) would be exactly zero. The local charges therefore have spins equal to the exponents of the Lie algebra modulo its Coxeter number h. No such pattern seems to exist for the supergroup model, and (2.30) seems to be non-zero for all positive integer values of r.

However after noting this, we can proceed with an analogous argument. We calculate $\{\ln A(x,\mu), \ln A(y,\nu)\}$ using (2.28), and thence it is seen that (2.31) is satisfied.

The infinite number of currents (2.30) are not algebraically independent. The number of independent currents is equal to the rank of the superalgebra, so for $su(m|n)$ there are $m+n-1$ independent currents. Therefore a maximal set of algebraically independent currents which form commuting charges are

$$\{K_i(x) \mid 2 \leq i \leq m+n\}. \tag{2.32}$$

Any higher spin currents which give further commuting charges must necessarily be algebraic functions of these currents. The form of the currents is identical to that of $\mathrm{SU}(m-n)$ [9], but we shall reproduce the first few examples of them here for completeness.

$$\begin{aligned}
\mathcal{K}_2 &= \mathcal{J}_2 \\
\mathcal{K}_3 &= \mathcal{J}_3 \\
\mathcal{K}_4 &= \mathcal{J}_4 - \frac{3}{2(m-n)}\mathcal{J}_2^2 \\
\mathcal{K}_5 &= \mathcal{J}_5 - \frac{10}{3(m-n)}\mathcal{J}_3\mathcal{J}_2
\end{aligned} \tag{2.33}$$

2.5 Commuting charges for $\mathrm{PSU}(n|n)$

To construct invariants on $psu(n|n)$, we first note that they are also necessarily invariants of $su(n|n)$. We can therefore consider invariants of $su(n|n)$ which are in addition invariant under the U(1) gauge symmetry $g(x) \mapsto e^{\phi(x)}g(x)$. Higher spin invariant currents of $su(n|n)$ are of the form $\mathcal{J}_m = \frac{1}{m!}\mathrm{Str}(j_\pm^m)$ for $j_\pm \in su(n|n)$. It is simple to show that

$$\mathcal{T}_2 = \mathcal{J}_2\ , \quad \mathcal{T}_{2m} = \sum_{k=2}^{2m-2} (-1)^k \mathcal{J}_k \mathcal{J}_{2m-k}\ , m \geq 2 \tag{2.34}$$

are invariant under the additional gauge symmetry.

Unfortunately our previous construction of charges in involution cannot be repeated for these models. There is no matrix representation of $psu(n|n)$ and we are forced to work in $su(n|n)$, where we encounter difficulties due to the degeneracy of the metric. We do not know how to circumvent this problem. Nevertheless, we still expect that an infinite number of commuting local charges exist, particularly as these models are quantum conformal. I hope to investigate this in future work.

2.6 Commuting charges for $\mathrm{OSp}(m|2n)$

Using the defining relation for this superalgebra, we see that $X \in osp(m|2n)$ implies that $X^p \in osp(m|2n)$ if p is odd. So for such p, we can use the completeness condition (1.16) to show that the integrand of (2.26) is a total divergence, and thus that the charges commute.

$$\{q_{\pm r}, q_{\pm s}\} = \pm\frac{2s(r+1)(s+1)\kappa}{r+s}\int dx\, \partial_x \mathrm{Str}(j_\pm^{r+s}) = 0 \tag{2.35}$$

So for the $\mathrm{OSp}(m|2n)$ model, the set of q_r defined by (2.25) for r odd is a set of mutually commuting charges. (For r even, these charges are exactly zero.)

We will find that we can derive more interesting conserved charges if we calculate the equal-time Poisson brackets for the currents using (2.12).

$$\{\mathcal{J}_r(x), \mathcal{J}_s(y)\} = -rs\mathcal{J}_{r+s-2}(x)\delta'(x-y) - \frac{rs(s-1)}{(r+s-2)}\mathcal{J}'_{r+s-2}(x)\delta(x-y) \tag{2.36}$$

This equation holds for all $r, s \geq 1$, but is only interesting for even r, s. Again we note the similarity between these relations and those given for the bosonic orthogonal and symplectic algebras [9]. Analogous to these models we will find a family of commuting currents, with a free parameter α. We can formulate these currents through the use of generating functions with a parameter λ. We define

$$B(x,\lambda) = \mathrm{sdet}(1 - \sqrt{\lambda} j_+(x)) = \exp\left(-\sum_{r=1}^{\infty} \frac{\lambda^r}{2r}\mathcal{J}_{2r}(x)\right) \tag{2.37}$$

and find that the currents defined by

$$\mathcal{K}_{r+1}(x) = B(x,\lambda)^{\alpha r}|_{\lambda^{(r+1)/2}} \tag{2.38}$$

give rise to a mutually commuting set of currents

$$\int dx\, dy\, \{B(x,\mu)^{\alpha r}, B(y,\nu)^{\alpha s}\}\Big|_{\mu^{(r+1)/2}\nu^{(s+1)/2}} = 0. \tag{2.39}$$

The argument proceeds similarly to the unitary case. The difference now is that we have a family of commuting charges, depending on a free parameter α. The resulting currents are similar in form to the orthogonal and symplectic cases [9]. We reproduce them here.

$$\begin{aligned}
\mathcal{K}_2 &= \mathcal{J}_2 \\
\mathcal{K}_4 &= \mathcal{J}_4 - \frac{3\alpha}{2}\mathcal{J}_2^2 \\
\mathcal{K}_6 &= \mathcal{J}_6 - \frac{15\alpha}{4}\mathcal{J}_4\mathcal{J}_2 + \frac{25\alpha^2}{8}\mathcal{J}_2^3 \\
\mathcal{K}_8 &= \mathcal{J}_8 - \frac{14\alpha}{3}\mathcal{J}_6\mathcal{J}_2 - \frac{7\alpha}{4}\mathcal{J}_4^2 + \frac{49\alpha^2}{4}\mathcal{J}_4\mathcal{J}_2^2 - \frac{343\alpha^3}{48}\mathcal{J}_2^4
\end{aligned} \tag{2.40}$$

2.6.1 The superpfaffian

For SO$(2l)$ there is another conserved current which cannot be expressed as the trace of a power of $j_+(x)$: the Pfaffian current of spin l,

$$\mathcal{P}(x) = \epsilon^{I_1 J_1 \ldots I_l J_l}(j_+)_{I_1 J_1} \ldots (j_+)_{I_l J_l} \tag{2.41}$$

By requiring that the charge associated to this current is in involution with the set of commuting charges formed from traces, the value of α is fixed to be $1/(2l-2)$ [9]. A similar situation exists for OSp$(2m|2n)$, for which there exists an analogous current, the superpfaffian current of spin $m-n$ [14]. Its charge is in involution with the supertrace currents fixes α to be $1/(2m-2n-2)$. The superpfaffian current differs from the others

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already considered in that it cannot be written in the form $d^{A_1\dots A_p} j_{\pm A_1}\dots j_{\pm A_p}$, but is instead a rational function.

Given any supermatrix written in block form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for which D is invertible, we define the superpfaffian to be

$$\mathrm{Spf}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Pfaff}(A - BD^{-1}C)}{\sqrt{\det(\mathrm{D})}} = \frac{\mathrm{Pfaff}(A)}{\sqrt{\det(D - CA^{-1}B)}} \tag{2.42}$$

where Pfaff is the ordinary pfaffian.

Let $j_+(x) \in osp(2m|2n)$. Every element of a Lie superalgebra is locally conjugate to some element of the Cartan subalgebra. So there exists $\mathrm{U}(x) \in \mathrm{OSp}(2m|2n)$ such that

$$Uj_+U^{-1} = \mathrm{diag}\left(\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{bmatrix}, i\mu_1, \dots i\mu_n, -i\mu_1, \dots, -i\mu_n\right) \tag{2.43}$$

for real $\lambda_i(x)$, $\mu_j(x)$ (the weights) [10]. The currents that we are interested in can be expressed as functions of the $m+n$ weights.

We first note that we can write (2.37) as

$$\mathrm{sdet}(1 - \sqrt{\nu} j_+(x)) = \frac{(1+\nu\lambda_1^2)\dots(1+\nu\lambda_m^2)}{(1+\nu\mu_1^2)\dots(1+\nu\mu_n^2)} \tag{2.44}$$

and then the supertrace currents (2.38) which give commuting charges can be expressed in terms of the weights as

$$\mathcal{K}_p(x) = \left[\frac{(1+\nu\lambda_1^2)\dots(1+\nu\lambda_m^2)}{(1+\nu\mu_1^2)\dots(1+\nu\mu_n^2)}\right]^{\alpha(p-1)}\Bigg|_{\nu^{p/2}} \tag{2.45}$$

or equivalently as

$$\mathcal{K}_p(x) = \exp\left(-\alpha(p-1)\sum_{r=1}\frac{(-1)^r\nu^r}{r}\left(\sum_i \lambda_i^{2r} - \sum_j \mu_j^{2r}\right)\right)\Bigg|_{\nu^{p/2}} \tag{2.46}$$

Our conserved superpfaffian current can then be written in terms of the weights as

$$\mathcal{P}(x) = \mathrm{spf}(j_+(x)) = \left|\frac{\lambda_1\dots\lambda_m}{\mu_1\dots\mu_n}\right| \tag{2.47}$$

and our claim is that requiring $\int dx\, dy\, \{\mathcal{K}_p(x), \mathcal{P}(y)\} = 0$ for all (even) p will constrain α to be $(2m-2n-2)^{-1}$. We shall use the Poisson bracket relations

$$\begin{aligned} \{\lambda_i(x), \lambda_j(y)\} &= -4\kappa\delta_{ij}\delta'(x-y) \\ \{\mu_k(x), \mu_l(y)\} &= 4\kappa\delta_{kl}\delta'(x-y) \end{aligned} \tag{2.48}$$

(where the difference in signs comes from the definition of supertrace) to calculate (writing $C(x) = \mathrm{sdet}(1-\sqrt{\nu}j_+(x))$ and $\beta = \alpha(p-1)$ for convenience)

$$\int dx\, dy\, \{\mathcal{P}(x), C(y)^\beta\} = \int dx\, dy\, \beta C(y)^{\beta-1}\{\mathcal{P}(x), C(y)\} \tag{2.49}$$

$$= 8\beta\kappa\nu \int dx \left(\sum_{i=1}^{m} \partial_x \left(\frac{\mathcal{P}(x)}{\lambda_i(x)} \right) \frac{C(x)^\beta \lambda_i(x)}{1+\nu\lambda_i^2(x)} - \sum_{k=1}^{n} \partial_x \left(\frac{\mathcal{P}(x)}{\mu_k(x)} \right) \frac{C(x)^\beta \mu_k(x)}{1+\nu\mu_k^2(x)} \right)$$
$$= 8\beta\kappa\nu \int dx \left[(m-n) - \frac{\nu}{\beta}\partial_\nu - 1 - \frac{1}{2\beta} \right] (\partial_x \mathcal{P}(x))\, C(x)^\beta \qquad (2.50)$$

We are only interested in the coefficient of $\nu^{p/2}$, and so we replace $\nu\partial_\nu \mapsto p/2-1$. We then find that the term in square brackets vanishes if and only if $\alpha = 1/(2m-2n-2)$, as required.

We immediately see that there is a similar pattern of spins for the local charges on $\mathrm{OSp}(m|2n)$ and $\mathrm{SO}(m-2n)$; Saleur and Kaufmann [12] have studied the similarity between the S-matrices of the models with these symmetries.

Interestingly, for the $\mathrm{OSp}(2n+2|2n)$ models (precisely those which are exactly conformal) there exists no finite value of α for which the superpfaffian charge commutes with all the other charges. This does not affect their integrability properties as it is still possible to construct a Lax pair.

3. Non-local charges

In addition to the local charges, there are two infinite sets of conserved non-local charges, which are elements of $(\mathbf{g}\otimes\Lambda)_{\bar{0}}$ and generate a chiral Yangian structure $Y(\mathbf{g})_L \times Y(\mathbf{g})_R$ [9, 28].

The full set of non-local charges are generated by the conserved local charge

$$Q_A^{(0)} = \int_{-\infty}^{\infty} dx\, j_{0A}(x) \qquad (3.1)$$

and the first non-local charge

$$Q_A^{(1)} = \int_{-\infty}^{\infty} dx j_{1A}(x) - \frac{1}{2\kappa} f^{BC}{}_A \int_{-\infty}^{\infty} dx\, j_{0B}(x) \int_{-\infty}^{x} dy\, j_{0C}(y) \qquad (3.2)$$

where conservation follows from (2.4). Higher charges are formed from commutations of these. Their construction in [28, 29] for the bosonic charges can be applied analogously here.

To show that all local charges are in involution with the non-local charges, it suffices to show that the local charges commute with the first two charges $Q_A^{(0)}$ and $Q_A^{(1)}$.

The invariance of the d-tensor can be used in a straightforward fashion to show that

$$\{q_s, Q_A^{(0)}\} = 0 \qquad (3.3)$$

Commutation of $Q_A^{(1)}$ is not so simple to show, but we can proceed by using a similar argument to [9]. We consider each of the two terms of $Q_B^{(1)}$ separately, and using the invariance property (2.16), we find that the commutation of the first term gives

$$\{q_s, \int dy\, j_{1B}(y)\} = -(s+1) \int dx\, d^{CA_1\dots A_s} f_C{}^D{}_B j_{+A_s} \cdots j_{+A_1} j_{1D} \qquad (3.4)$$

As in [9], when looking at the second term we must be cautious when working with the limits of the spatial integration. We thus integrate between $\pm L$ and then take the limit $L \to \infty$. We are interested in

$$\{q_s, \int_{-L}^{L} dy \int_{-L}^{y} dz\, f^{CD}{}_B j_{0C}(y) j_{0D}(z)\} \tag{3.5}$$

To evaluate this, we then introduce a step function,

$$\int_{-L}^{L} dx \int_{-L}^{L} dy \int_{-L}^{L} dz\, d^{A_1 \ldots A_{s+1}} f^{CD}{}_B \{j_{+A_{s+1}}(x) \ldots j_{A_1}(x), j_{0C}(y) j_{0D}(z)\} \theta(y-z) \tag{3.6}$$

All the ultra-local terms (i.e. the $\delta(x-y)$ and $\delta(x-z)$) vanish by invariance (2.16), and after some computation (noting that the currents vanish at infinity) we are left with

$$2\kappa(s+1) \int dx\, d^{CA_1 \ldots A_s} f_C{}^D{}_B j_{+A_s} \ldots j_{+A_1} j_{0D} \tag{3.7}$$

We recombine results (3.4) and (3.7), and again use the invariance property (2.16) to show the final result

$$\{q_s, Q_A^{(1)}\} = 0 \tag{3.8}$$

The Yangian charges do not in general commute.

$$\{Q_A^{(0)}, Q_B^{(0)}\} = (-1)^\chi f_{AB}{}^C Q_C^{(0)} \tag{3.9}$$

$$\{Q_A^{(0)}, Q_B^{(1)}\} = (-1)^\chi f_{AB}{}^C Q_C^{(1)} \tag{3.10}$$

where $\chi = \eta_A . \eta_B + \eta_A + \eta_B$. Once equipped with the additional structure of a (non-trivial) coproduct and counit, this is the expected form of the super-Yangian algebra [16 − 18].

4. Remarks on the quantum model

To determine whether the higher spin local currents are also conserved in the quantum theory, we use the anomaly counting method of Goldschmidt and Witten [15]. This is a rather indirect method, and does not convey an insight into the form of these anomalies. Instead it tries to show that any quantum anomalies can be written in the form of a total derivative, in which case a conservation law still exists, although in a modified form.

We must consider all possible anomaly terms which have the same behaviour under the symmetries of the model. These symmetries comprise of the continuous Lorentz and chiral symmetries, and also some discrete symmetries.

For any supergroup, the principal chiral model is invariant under the map $\pi : g \mapsto g^{-1}$. This exchanges the left and right currents. Additional symmetries arise as outer autmorphisms of the underlying Lie superalgebras [19].

$$\gamma : g \mapsto g^* \quad \text{for } \mathbf{g} = su(m|n) \tag{4.1}$$

$$\sigma : g \mapsto MGM^{-1} \text{ for } \mathbf{g} = osp(2m|2n) \tag{4.2}$$

where M is an element of $\mathrm{OSp}(2m|2n)$ with superdeterminant -1. Each of the currents $\mathcal{J}_r(x)$ are either odd or even under the action of all of these symmetries. The argument proceeds identically to the bosonic PCM [9], and we shall not reproduce it all here, but instead illustrate the idea with the spin 2 example. It does not matter for the following whether we are considering $su(m|n)$ or $osp(m|2n)$; the results are the same for all models.

There are only two spin 2 currents, $\mathcal{J}_2 = \mathrm{Str}(j_\pm^2)$. (We shall just consider the + current, the argument is identical for the other.) This is even under the discrete symmetries, invariant under the chiral symmetry, and of mass dimension 2. There is only one possible anomaly term with identical behaviour, $\mathrm{Str}(j_-\partial_+ j_+)$, which can be written as a total derivative $\partial_+\mathrm{Str}(j_- j_+)$. So the only possible correction to the conservation law is

$$\partial_- \mathcal{J}_2 = \alpha \partial_+ \mathrm{Str}(j_- j_+) \tag{4.3}$$

where α is some unknown parameter, and we have a modified conservation law. (Note that α can be zero, and we would expect it to be so for those models which retain quantum conformal invariance.)

Similar results hold for spin 3 and 4 currents, but not for higher spins. This does not mean that there is no quantum conserved current, for the anomaly counting method is sufficient but not necessary. Nevertheless, integrability is guaranteed by the existence of one higher spin conservation law [30].

In the quantum model, the local charges are not enough to give the particle multiplets, and we must regard the particle states to be in a representation $(V, \bar{V})$ of the chiral super-Yangian $Y(\mathbf{g})_L \times Y(\mathbf{g})_R \supset \mathbf{g}_L \times \mathbf{g}_R$ [9, 17]. Representations of superalgebras differ considerably from the bosonic algebra, particularly the issue of atypicality [19].

We hope to pursue these issues further, by considering the S-matrix of these models [12]. It is anticipated that all the particle states are formed through tensor products of basic representations of $Y(\mathbf{g})$ (the bootstrap programme). Unlike bosonic algebras, the tensor product of two irreducible representations of a superalgebra need not be reducible (and indeed is only so for $osp(2|2n)$). We have seen that, for all superalgebras considered, the super-Yangian commutes with each of the local charges classically. If this holds for each local charge which survives quantization, then every particle multiplet will have the same charge number, and their pattern of values should provide an alternative source of information about the bootstrap programme.

5. Conclusions and further questions

We have derived a set of commuting charges for the principal chiral model on the Lie supergroups $\mathrm{SU}(m|n)$ for $m \neq n$, and $\mathrm{OSp}(m|2n)$. These are integrals of local currents, each constructed with the use of an invariant of the underlying Lie superalgebra.

The $\mathrm{SU}(m|n)$ models have conserved currents generated by $m+n-1$ primitive currents. The current algebra is similar in appearance to that of $\mathrm{SU}(m-n)$.

The orthosymplectic models $\mathrm{OSp}(2m|2n)$ and $\mathrm{OSp}(2m+1|2n)$ each have $m+n$ primitive currents from which the infinite charges can be calculated. For the former set of models, the superpfaffian charge will only commute with the other charges if $m \neq n+1$.

The algebra of currents suggests a relationship between $\mathrm{OSp}(m|2n)$ and $\mathrm{SO}(m-2n)$ models [12].

In the bosonic model there is a correlation between the degrees of the primitive currents and the exponents of the underlying Lie algebra [9]. This same pattern of currents exists for the affine Toda field theories [31 − 34]. It is therefore natural to wonder whether a similar pattern exists between the local charges of the PCM on a supergroup and those of the affine Toda field theory on a Lie superalgebra [35, 36]. The difficulty here is that to each Lie superalgebra there may be associated more than one affine Toda field theory, depending on the choice of inequivalent simple root system. It would be interesting to consider whether the PCM shares any properties with these models.

Quantum conformal invariance is not expected for general Lie supergroups, but it should be possible to introduce a WZ term which guarantees this at a certain critical limit. The integrability of these models through a consideration of their local and non-local conserved charges will be the scope of future work.

Similar to the arguments in this paper, the sigma model on a supercoset (either with or without a WZ term) should exhibit local conserved charges. The construction of a set of (classically) commuting charges for these models has yet to be explored.

For the PCM on a Lie Group in the presence of a boundary (i.e. on the half-line $x < 0$) a natural connection arises between boundary integrability and symmetric spaces [37]. Future work will investigate the analogous boundary conditions to ensure integrability on a supergroup.

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